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Limit Theorems of Integrals with Respect to Vector Random Measures in Complete Paranormed Spaces

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Abstract: This paper studies random integral of the form $\int gdM$, where f is a function taking value in a paranormed vector space X, and M is an independent scattered vector random measure. Random integrals of this type are a natural generalization of random series with paranormed space valued coefficients. Some limit theorems of integrals with respect to vector random measures are proved.

Keywords: Paranormed Vector Space, Random Measure, Random Integral, Limit Theorem, Convergence in Probability

1. Introduction

The idea of random measures first appeared in Bochner's paper [1], and a variety of discussions followed [8, 10, 12, 15, 17, 18, 19, 20]. The aim of this article is to study random integral of the form $\int gdM$, where g is a function taking value in a paranormed vector space X, and M is an independent scattered vector random measure. Random integrals of this type are a generalization of random series with Banach space valued coefficients. It is well known that the asymptotic behavior of such series depends also on some geometric properties of the Banach space X (or a metrizable space) [3, 4, 6, 8, 16]. On the other hand, the existence of certain bounded linear operators on appropriate function spaces which we call random integrals depends in general on a geometric structure of X [12, 18, 19].

This paper defines and studies random integrals without any restriction in the geometry of X, devotes to the study of independently scattered vector random measures with emphasis on their convergence properties, presents convergence theorems of random integrals of the forms $\int g_n dM$ and $\int gdM$, where g_n and g are taking values in a paranormed vector space X, and M_n and M are R^1-valued random measures.

2. Paranormed Spaces

Let K be the field of real numbers or the field of complex numbers, and X be a vector space over K. A paranorm $\|\cdot\|$ is a function defined on X, satisfying

$$\| x \|\geq 0, \ \| x \|= 0 \Leftrightarrow x = 0;$$

$$\| x + y \|\leq\| x \| + \| y \|;$$

$\| -x \|=\| x \|$; and

$\lim_{n \to 0} \| \alpha_n x \|= 0$, for any sequence $\{\alpha_n\} \subset K$;

$\lim_{n \to 0} \| \alpha x_n \|= 0$, for any sequence $\{x_n\} \subset X$.

Since $\| x - y \|\leq\| x \| + \| y \|$, for $\forall x, y \in X$, $\| x - y \|$ defines a metric on X. X equipped with the metric topology is said to be a paranormed space.

A paranormed space is said to be complete if every Cauchy sequence $\{x_n\}$ is convergent in X.

Any Banach space is a complete paranormed space. But the converse is not true.

A seminorm is function q defined on X and satsfying
$q(x) \geq 0;$
$q(x+y) \leq q(x) + q(y);$ and
$q(\alpha x) = |\alpha| q(x)$ for any scalar α.

From [2], one has the following Lemma 1.1.

Lemma 1.1 X is a complete paranormed space if and only if there is a family of continuous siminorms $Q = \{q_{n;}\ n = 1, 2, \ldots.\}$ on X, such that

$$q_1(x) \le q_2(x) \le \cdots \le q_n(x) \le \cdots,\ \forall x \in X\ .$$

And the paranorm on X can be given by

$$\|x\| = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{q_n(x)}{1 + q_n(x)},\ \forall x \in X\ .$$

For any sequence $\{x_n\} \subset X$, and $x \in X$, the following are equivalent

$$\lim_{n \to \infty} x_n = x\ ;$$

$$\lim_{n \to \infty} \|\ x_n - x\ \| = 0\ ;\ \text{or}$$

$$\lim_{n \to \infty} q(x_n - x) = 0\ ,\ \forall q \in Q\ .$$

3. Random Integrals

Let (T, Σ) be a measurable space, and (Ω, Γ, P) be a probability space. A function

$$M\colon \Sigma \to L_0(\Omega, \Gamma, P)$$

is said to be an independently scattered random measure if for every pairwise disjoint sets $A_n \in \Sigma$, random variable $M(A_n)$ are independent, $n = 1, 2, \ldots,$ and $M(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} M(A_n)$.

Every independently scattered random measure M can be decomposed into two independent and independently scattered random measure $M = M_a + M_c$, where M_a is pure atomic, and M_c is atomless.

Let (T, Σ, μ) be a finite measure space, and ν be an infinitely divisible distribution on R^1. The function

$$M\colon \Sigma \to L_0(\Omega, \Gamma, P)$$

is a random measure on (T, Σ, μ) generated by ν if M is independently scattered and, for every $A \in \Sigma$,

$$\mathscr{L}(M(A)) = \nu^{*\mu(A)}\ ,$$

where ν^{*p} is the p$^{\text{th}}$ convolution power of ν.

Daniell-Kolmogorov's Consistency Theorem implies the existence of a random measure on every measure space generated by every infinite divisible law.

In what follows we assume that ν is symmetric. From [3], we may write

$$E \exp(iuM(A)) = \exp[-\mu(A)K(u)]\ ,\qquad (1)$$

where

$$K(u) = \frac{1}{2}\sigma^2 u^2 + \int_{-\infty}^{+\infty} (1 - \cos uv)m(dv)\ ,\qquad (2)$$

while m is a symmetric σ-finite measure on R^1 such that $M(\{0\}) = 0$ and $\int_{-\infty}^{+\infty} \min(1, v^2)m(dv) < \infty$. We write $M \cong [\sigma^2, m]$ if (1) and (2) hold.

In the sequel let X be a paranormed space, X' be the topological dual space of X.

A function $g\colon T \to X$ is said to be a simple function if there exist pairwise disjoint measurable sets $B_j \in \Sigma$ and $x_j \in X$ ($j = 1, 2, \ldots, k$) such that

$$g(s) = \begin{cases} x_j, & if \quad s \in B_j \ (j = 1, \cdots, k) \\ 0, & otherwise \end{cases}$$

For every $A \in \Sigma$, we set

$$\int_A gdM = \sum_{j=1}^{k} x_j M(B_j \bigcap A)\ ,$$

the μ- Integral of the simple function g is defined as

$$\int_T g(s)d\mu = \sum_{j=1}^{k} x_j \mu B_j$$

Definition 3.1 A function $g\colon T \to X$ is said to be M-Integrable, if there exists a sequence of $\{g_k\}$ of simple functions from T to X such that

$$\lim_{k \to \infty} g_k(s) = g(s)\ ,\ \ \mu\text{-a.e. on } T \qquad (3)$$

i.e., there exists $E \in \Sigma$ and $\mu(E) = 0$ such that $\lim_{k \to \infty} g_k(s) = g(s)$ for $s \in T \setminus A$; and for every $A \in \Sigma$, $\int_A g_k dM$ converges in probability.

If g is M-integrable, we put

$$\int_A gdM = P - \lim_{k \to \infty} \int_A g_k dM\ ,$$

then the symmetry and independence assumptions imply that for every seminorm $q \in Q$, every simple function $g\colon T \to X$, and every $A \in \Sigma$

$$P\{q(\int_A g_k dM) > \varepsilon\} \le P\{q(\int_T g_k dM) > \varepsilon\}, \varepsilon > 0$$

Hence condition (3) in the Definition 3.1 is equivalent to

$$\int_T g_k dM\ \text{ converges in probability.}\qquad (4)$$

Let $\mathscr{L}_E(M)$ denote the linear subspace of $L_0^X(T, \Sigma, \mu)$ consisting of all M-integrable functions. Similar to the proof of the Theorem in [5], $\mathscr{L}_E(M)$ is a complete metrizable vector space with the paranorm

$$\| g \|_M = \int_T \min\{1, \| g \|\}\mu(dt) + E \min\{1, \| \int_T fdM \|\}$$

A random variable $M \cong [\sigma^2, m]$ is said to be Poissonian if $\sigma^2 = 0$ and $M(R^1) < \infty$.

Proposition 3.1 If M is a poissonian random measure on (T, Σ, μ), then

$$\mathcal{L}_E(M) = L_0^X(T, \Sigma, \mu)$$

Proof. Without loss of generality, we may assume that $M(R^1) > 0$. For $u \geq 0$, set

$$H(u) = \int \min\{u | v |, 1\} m(dv)$$

then

$$E \min\{u | M(A) |, 1\}$$
$$= e^{-\mu(A)m(R)} \sum_{k=0}^{\infty} \frac{\mu^k(A)}{k!} \int_{-\infty}^{+\infty} \min\{u | v |, 1\} m^{*k}(dv)$$
$$\leq H(u)\mu(A).$$

Therefore, for any simple function

$$g(s) = \begin{cases} x_j, & if \quad s \in B_j \quad (j = 1, \cdots, k) \\ 0, & otherwise \end{cases}$$

where $B_j \in \Sigma$ pairwise disjoint measurable sets, and for any seminorm q in Q we have

$$E \min\{q(\int_T f dM), 1\}$$
$$\leq \sum_{j=1}^{k} E \min\{q(x_j) | M(A_j) |, 1\}$$
$$\leq \sum_{j=1}^{k} H(q(x_j))\mu(A_j)$$
$$= \int_T H(q(f))\mu(dt).$$

The proof is completed.

Proposition 3.2 Suppose that $E | M(T) | < \infty$. Then, there exists a constant C_M such that for every simple function $g: T \to R^1$,

$$E | \int_T g dM | \leq c_M (\int_T g^2 d\mu)^{1/2}$$

Proof. Suppose $M = M_a + M_c$, where M_a is pure atomic, and M_c is atomless.

Write $g(s) = \sum_{n=1}^{k} b_n \chi_{B_n}$, where $B_n \in \Sigma$ pairwise disjoint measurable sets, one has

$$E | \int_T g dM_a |$$
$$\leq (E | \int_T g^2 dM_a |)^{1/2}$$
$$= (\sum_{n=1}^{k} b_n^2 E M_a^2(B_n))^{1/2}$$
$$= c_0 (\int_T g^2 d\mu |)^{1/2},$$

where c_0 is a constant. And there exists a constant c_1 such that

$$E | \int_T g dM_c |$$
$$\leq (\sum_{n=1}^{k} | b_j | E M_1(B_n))^{1/2}$$
$$\leq c_1 \int_T | g | d\mu |.$$

Taking $c_M = c_0 + c_1 \mu^{1/2}(T)$, then

$$E | \int_T g dM | \leq c_M (\int_T g^2 d\mu)^{1/2}$$

Proposition 3.3 If g is M-integrable, $x' \in X'$, then $x'(g)$ is M-integrable, and

$$x'(\int_A g dM) = \int_A x'(g) dM$$

Proof. The M-integrability implies that there exists a sequence $\{g_k\}$ of simple functions from T to X such that

$$\lim_{k \to \infty} g_k(s) = g(s), \quad \mu\text{-a.e. on } T$$

and

$$\int_T g dM = P - \lim_{k \to \infty} \int_T g_k dM$$

Noting that g_k can be written as

$$g_k(s) = \sum_{n=1}^{n(k)} x_n \chi_{B_{n,k}}$$

where $\chi_{B_{n,k}}$ is the characteristic function of the set $B_{n,k}$, i.e.,

$$\chi_{n,k}(s) = \begin{cases} 1 & s \in B_{n,k} \\ 0, & otherwise \end{cases}$$

Therefore

$$x'(g_k(s)) = \sum_{n=1}^{n(k)} x'(x_n) \chi_{B_{n,k}}$$

Hence

$$\lim_{k \to \infty} x'(g_k(s)) = \lim_{k \to \infty} \sum_{n=1}^{n(k)} x'(x_n) \chi_{B_{n,k}} = x'(g_k(s))$$

μ-a.e. on T. And

$$P - \lim_{k \to \infty} \int_T x'(g_k) dM$$
$$= P - \lim_{k \to \infty} \sum_{n=1}^{n(k)} x'(x_n) M(B_{n,k})$$
$$= x'(P - \lim_{k \to \infty} \sum_{n=1}^{n(k)} x_n M(B_{n,k}))$$
$$= x'(P - \lim_{k \to \infty} \int_T g_k dM)$$
$$= x'(\int_T g dM).$$

Consequently, $x'(g)$ is M-integrable. And by the uniqueness of the limit one concludes that

$$x'(\int_A gdM) = \int_A x'(g)dM$$

4. Limit Theorems

This section devotes to the proofs of some limit theorems of integrals with respect to vector random measures.

A sequence $\{x_k\} \subset X$ is said to be weakly convergent to $x \in X$ if for every $x' \in X'$, there holds $\lim_{k\to\infty} x'(x_k) = x'(x)$, and denoted by $w - \lim_{k\to\infty} x_k = x$.

Definition 4.1 A probability measure μ is said to be a control measure of the random vector measure M, and denoted by $M \ll \mu$, if $\mu(A) = 0$ implies $M(A) = 0$.

Definition 4.2 A sequence $\{M_k\}$ of random vector measures is said to be weakly converges in probability to a random vector measure M, denoted by $wP - \lim_{k\to\infty} M_k = M$, if for every $x' \in X'$, every $E \in \Sigma$, there holds

$$P - \lim_{k\to\infty} x'(M_k(E) - M(E)) = 0$$

Definition 4.3 A function $g: T \to X$ is said to be weakly M-integrable, if for every $x' \in X'$, $x'(g)$ is M-integrable.

Theorem 4.1 Suppose $M \ll \mu$. $g, g_k \in \mathscr{L}_E(M)$ ($k = 1, 2,\ldots$), such that $w - \lim_{k\to\infty} g_k(s) = g(s)$ μ-a.e. If for any given $t > 0$, and if for any $x' \in X'$, and any $A \in \Sigma$

$$\lim_{\mu(A)\to 0} \sup_k P\{\| x'(\int_A g_k dM) \| > t\} = 0$$

then

$$wP - \lim_{k\to\infty}\int_A g_k dM = \int_A gdM$$

Proof. From Egoroff Theorem, for any given $\varepsilon > 0$, there exists $A(\varepsilon) \in \Sigma$, $A(\varepsilon) \subset A$ and there exists $N > 0$, such that $\mu(A \setminus A(\varepsilon)) < \varepsilon$, and for all $s \in A(\varepsilon)$

$$\| g_k(s) - g(s) \| < \varepsilon, k > N \qquad (5)$$

Hence

$$P[\| x'(\int_{A(\varepsilon)} g_k dM - \int_{A(\varepsilon)} gdM) \| > t]$$
$$= P[\| x'(\int_{A(\varepsilon)} x'(g_k - g)dM) \| > t] \qquad (6)$$
$$\leq 2P\{\| x'(g_k - g) \|_\infty \cdot \| M(T) \|\},$$

Combining (5) and (6) one concludes that

$$\lim_{k\to\infty} P[\| x'(\int_{A(\varepsilon)} g_k dM - \int_{A(\varepsilon)} gdM) \| > t] = 0$$

Therefore, for any given $x' \in X'$

$$P[\| x'(\int_A g_k dM - \int_A gdM) \| > t]$$
$$\leq P[\| x'(\int_A g_k dM - \int_{A(\varepsilon)} g_k dM) \| > t]$$
$$+ P[\| x'(\int_{A(\varepsilon)} g_k dM - \int_{A(\varepsilon)} gdM) \| > t]$$
$$+ P[\| x'(\int_{A(\varepsilon)} gdM - \int_A gdM) \| > t].$$

$$= P[\| x'(\int_{B(\varepsilon)} g_k dM) \| > t]$$
$$+ P[\| x'(\int_{A(\varepsilon)} g_k dM - \int_{A(\varepsilon)} gdM) \| > t]$$
$$+ P[\| x'(\int_{B(\varepsilon)} gdM) \| > t],$$

where $B(\varepsilon) = A \setminus A(\varepsilon)$. From the assumption

$$\lim_{k\to\infty} P[\| x'(\int_{B(\varepsilon)} g_k dM) \| > t] = 0$$
$$\lim_{k\to\infty} P[\| x'(\int_{B(\varepsilon)} gdM) \| > t] = 0$$

Therefore

$$wP - \lim_{k\to\infty}\int_A g_k dM = \int_A gdM$$

Similar to the proof of Theorem 4.1 one has the following Theorem 4.2.

Theorem 4.2 Suppose $M \ll \mu$. $g, g_k \in \mathscr{L}_E(M)$ ($k = 1, 2,\ldots$), such that $\lim_{k\to\infty} g_k(s) = g(s)$ μ-a.e. For given $t > 0$, if

$$\lim_{\mu(A)\to 0} \sup_k P\{\| x'(\int_A g_k dM) \| > t\} = 0$$

Then, for any $A \in \Sigma$

$$P - \lim_{k\to\infty}\int_A g_k dM = \int_A gdM$$

Theorem 4.3 Suppose $M \ll \mu$. $g, g_k \in \mathscr{L}_E(M)$ ($k = 1, 2, \ldots$), such that $\lim_{k\to\infty} g_k(s) = g(s)$ μ-a.e. If there exists $C > 0$ such that $\| g_k(s) \| \leq C, (n = 1, 2, \ldots)$, then g is M-integrable, and for every $A \in \Sigma$

$$P - \lim_{k\to\infty}\int_A g_k dM = \int_A gdM.$$

Proof. Since $M \ll \mu$,

$$\lim_{\mu(A)\to 0} \sup_k P[\| \int_A g_k dM \| > t]$$
$$\leq \lim_{\mu(A)\to 0} \sup_k 2P[\| g_k \|_\infty \cdot \| \int_A 1dM \| > t]$$
$$\leq \lim_{\mu(A)\to 0} \sup_k 2P[C \cdot \| M(A) \| > t]$$
$$= 0.$$

Therefore, from Theorem 4.2

$$P - \lim_{k\to\infty}\int_A g_k dM = \int_A gdM$$

Theorem 4.4 Suppose $M, M_k (k = 1,2,...)$ are random vector measures, and $P - \lim_{k \to \infty} M_k = M$, $g \in \mathscr{L}_E(M)$ and g is a bounded function under the weak topology of X, then g is weakly M-integrable and for every $A \in \Sigma$

$$wP - \lim_{k \to \infty} \int_A g dM_n = \int_A g dM$$

Proof. From Proposition 2.3, g is weakly M-integrable. Noting that $P - \lim_{k \to \infty} M_k = M$

$$P[\| x'(\int_A g dM_k - \int_A g dM) \| > t]$$
$$\leq P[\| \int_A x' g(dM_k - dM) \| > t]$$
$$\leq 2P[\| x'(g) \|_\infty \cdot \|(M_k - M)(T)\| > t]$$
$$+ P[\| x'(\int_{A(\varepsilon)} g dM - \int_A g dM) \| > t].$$
$$= 0.$$

This completes the proof.

Theorem 4.5 Suppose $M, M_k (k = 1,2,...)$ are random vector measures, where $M_k \cong [\sigma^2, \mu_n]$, and $E | M_k(T)| < \infty$ ($k = 1, 2, ...$), g is a bounded continuous function. If $\{\mu_k\}$ weakly converges to μ, and if $\{M_k\}$ weakly converges to M, then

$$wP - \lim_{k \to \infty} \int_T g dM_n = \int_T g dM$$

Proof. For every $n > 0$, there exists a compact subset $B \subset T$ and a constant $c > 0$ such that

$$\sup_k \mu_k(B) < c < \infty, \quad \sup_k \mu_k(T \setminus B) < \frac{1}{n},$$

and there exists a sequence of simple function

$$g_l(s) = \sum_{n=1}^{n(l)} x_n \chi_{B,l} \quad (l = 1, 2, ...)$$

such that

$$\| g_l(s) - g(s) \| < \frac{1}{n}, \quad s \in B,$$

and

$$\sup_s \| g_l(s) \| \leq \sup_s \| g(s) \| < c_0 \ (l = 1, 2, ...).$$

Therefore

$$\int_T \| g - g_l \|^2 d\mu_n$$
$$= \int_B \| g - g_l \|^2 d\mu_k + \int_{T \setminus B} \| g - g_l \|^2 d\mu_k$$
$$\leq n^{-2} \mu_n(B) + (2c_0)^2 \mu_n(T \setminus B)$$
$$\leq n^{-2} c + (2c_0)^2 n^{-1}.$$

Hence, there exists a constant $c_1 > 0$ such that

$$E \| \int_T g_l dM_k - \int_T g dM_k \|$$
$$\leq c_1 (m^{-2} c + (2c_0)^2 n^{-1})^{\frac{1}{2}},$$

By Chebyshev's Inequality, for every $t > 0$

$$\lim_{l \to \infty} \lim_k \sup P\{\| \int_T g_l dM_k - \int_T g dM_k \| > t\} = 0$$

Similarly,

$$\lim_{l \to \infty} P\{\| \int_T g_l dM - \int_T g dM \| > t\} = 0$$

Noting that

$$\lim_k \sup P\{\| \int_T g dM_k - \int_T g dM \| > t\}$$
$$\leq \lim_k \sup P\{\| \int_T g dM_k - \int_T g_l dM_k \| > t\}$$
$$+ \lim_k \sup P\{\| \int_T g_l dM_k - \int_T g_l dM \| > t\}$$
$$+ \lim_k \sup P\{\| \int_T g_l dM - \int_T g dM \| > t\}$$
$$= \lim_k \sup P\{\| \int_T g_l dM_k - \int_T g_l dM \| > t\}$$
$$+ \lim_k \sup P\{\| \int_T g_l dM - \int_T g dM \| > t\}.$$

Hence

$$\lim_{k \to \infty} P\{\| \int_T g dM_k - \int_T g dM \| > t\} = 0$$

That is to say

$$P - \lim_{k \to \infty} \int_T g dM_n = \int_T g dM$$

On the other hand, for any $A_n \in \Sigma$ ($n = 1, 2,...$) satisfying $\lim_{n \to \infty} \mu(A_n) = 0$ there exists a constant $c_2 > 0$ such that

$$E \| \int_{A_n} g dM_k \| \leq c_2 (\int_{A_n} \| g \|^2 d\mu_k)^{\frac{1}{2}}$$

Since $\{\mu_k\}$ weakly converges to μ,

$$\lim_{n \to \infty} \lim_k \sup \int_{A_n} \| g \|^2 d\mu_k \leq \int_{A_n} \| g \|^2 d\mu$$

Hence

$$\lim_{n \to \infty} \lim_k \sup E \| \int_{A_n} g dM_k \| = 0$$

Again, Chebyshev's Inequality implies that

$$\lim_{n \to \infty} \lim_k \sup P\{\| \int_{A_n} g dM_k \| > t\} = 0$$

Which completes the proof.

5. Conclusion Remark

Let (T, Σ) be a measurable space. A countably additive set function on Σ and taking values in $L_0(\Omega, P)$ is said to be a random measure. Vector random measures arise naturally as a "Banach space generalization" of real-valued random measures. Let X be a Banach space, $L_0^X(\Omega, P)$ the set of all X-valued random variables. A countably additive set function on Σ taking values in $L_0^X(\Omega, P)$ is said to be an X-valued random measure. Vector random measures can be regarded as a "natural random generalization" of vector non-random measures studied by many authors.

This paper extends the definitions of random measure and random integral to complete paranormed spaces, and devotes to the study of limit theorems of random integrals with respect to vector random measures.

References

[1] S. Bochner, Stochastic Process, Annals of Mathematics, (48)1947(1), 1014-1061.

[2] F. Treves, Topological Vector Spaces, Distribution and Kernels, Academic Press, 1967.

[3] J. Rosinski, Random Integrals of Banach Space Valued Functions, Studia Mathematica, 8(1984), 15-38.

[4] P. Billingsley, Convergence of Probability Measures, John Wiley & Son, Inc. 1968.

[5] R. Zeng, the Completeness of $L_X^1(\mu, \Sigma)$, Journal of Mathematical Research with Applications, 15(1995)5, 40-48.

[6] L. Egghe, The Radon-Nikodym Property, densibility and Martingles in Loccally Convex Space, Pacific Journal of Mathematics, 87(1980)2, 313-322.

[7] D. H. Thang, On the Convergence of Vector Random Measures, Probability Theory and Related Fields, 88(1991), 1-16.

[8] A. N. Baushev, On the Weak Convergence of Probability Measures in Orlicz Spaces, Theory of Probability and Applications, 40(1996)3, 420–429.

[9] R. Jajte and W. D. A. Paszkiewicz, Probability and Mathematical Statistics Conditioning and Weak Convergence, Probability and Mathematical Statistics, 19(1999)2, 453-461.

[10] N. Guillotin-Plantard and V. Ladret, Limit Theorems for U-statistics Indexed by a One Dimensional Random Walk, ESAIM: Probability and Statistics, 9(2005), 98-115.

[11] H. Tsukahara, On the Convergence of Measurable Processes and Prediction Processes, Illinois Journal of Mathematics, 51(2007)4, 1231–1242.

[12] X. F. Yang, Integral Convergence Related to Weak Convergence of Measures, Applied Mathematical Sciences, 56(2011)5, 2775 – 2779.

[13] T. Grbic and S. Medic, Weak Convergence of Sequences of Distorted Probabilities, SISY 2015, Proceedings of IEEE 13th International Symposium on Intelligent System and Informatics, Sept. 2015, 307-318, Subotica, Serbia.

[14] T. E. Govindan, Weak Convergence of Probability Measures of Yosida Approximate Mild Solutions of McKean-Vlasov Type Stochastic Evolution Equations, Seventh International Conference on Dynamic Systems and Applications & Fifth International Conference on Neural, Parallel, and Scientific Computations, May 2015, Atlanta, USA.

[15] P. Puchala, Weak Convergence in L^1 of the Sequences of Monotonic Functions, Journal of Applied Mathematics and Computational Mechanics 13(2014)3, 195-199.

[16] L. Meziani, A Theorem of Riesz Type with Pettis Integrals in Topological Vector Spaces, Journal of Mathematical Analysis and Applications, 340 (2008), 817–824.

[17] F. J. Pinski, G. Simpson, A. M. Stuart and H. Weber, Kullback-Leibler Ppproximation for Probability Measures on Infinite Dimensional Spaces, SIAM Journal on Mathematical Analysis, 47(2015)6, 4091-4122.

[18] W. Lohrl and T. Ripple, Boundedly Finite Measures: Separation and Convergence by an Algebra of Functions, Electronic Communication of Probability, 21(2016)60, 1–16.

[19] H. H. Wei, Weak Convergence of Probability Measures on Metric Spaces of Nonlinear Operators, Bulletin of the Institute of Mathematics Academia Sinica, 11(2016)3, 485-519.

[20] D. Dolgopyat, A Local Limit Theorem for Sums of Independent Random Vectors, Electronic Journal of Probability, 39(2017)1–15.

[21] G. Naitzat and R. J. Adler, A Central Limit Theorem for the Euler Integral of a Gaussian random field, Stochastic Processes and Their Applications, 127(2017)6, 2036-2067.