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Quartic Transmuted Exponential Distribution: Characteristics and Parameter Estimation

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Abstract: The scope for generating high-rank transmuted distributions has expanded beyond the cubic to achieve improved performance in baseline distributions such as those of the Gamma type. This paper develops a Quartic Rank Transmutation Distribution (QRTD), a new family of transmuted distributions with enhanced flexibility for modelling complex data problems, including those with multi-modal distributions. Application is carried out to obtain a transmuted exponential distribution (QTED). Various characteristics of the new exponential distribution are presented, including the cumulative distribution function, the reliability and hazard functions, moments, and relevant order statistics. These features support the legitimacy and robustness of the proposed QTED. Additionally, the paper identifies specific parameter ranges that exhibit notable behaviours in the new distribution and its survival quantities. The maximum likelihood estimates of parameters are described, with simulation studies indicating that their precision improves with larger sample sizes. The performance of the QTED is found to be superior to existing lower-rank cubic and quadratic transmuted exponential distributions based on information criteria using real lifetime data. The applications demonstrate that the high-rank transmutation map could be instrumental in obtaining new transmutations of other relevant distributions with improved performance. This development signifies a major advancement in the field of probability distributions, offering more sophisticated tools for statisticians and researchers to model and analyse complex data patterns more accurately and effectively. Thus, the QRTD and its applications hold significant promise for future research and practical implementations in various statistical and applied fields.

Keywords: Quartic Transmutation, Transmuted Exponential Distribution, Parameter Estimation, Order Statistics

1. Introduction

The growing interest in the development of statistical distributions that are more flexible for modelling data remains prominent, especially in the context of lifetime distributions. The literature contains several extensions and modifications of well-known lifetime distributions [12, 4, 9, 7, 6]. These extensions and modifications are proposed to serve as useful methods to model a wide range of real lifetime phenomena across various domains.

Transmutation maps, introduced by [15], have become one of the most adopted approaches to extend or modify statistical distributions. This approach is proposed as a pragmatic means of constructing new and more flexible probability distributions. Quadratic and cubic rank transmutation maps have been studied in the literature. This paper contributes to the ongoing studies in the area of transmuted distributions by means of a higher rank transmutation map.

The new parametric family introduces distributions that are not only analytically tractable but also adept at accommodating complex datasets, including those exhibiting bimodal distribution or bimodal hazard rates. The emphasis of this study is particularly directed towards the application of quartic rank in the context of the exponential distribution. Shaw & Buckley [15] developed a new family of probability distributions using the quadratic rank transmutation map and called it the transmuted distribution. The cdf of the quadratic transmuted distribution is defined as:

$$G(x) = (1+\lambda)F(x) - \lambda F^2(x) \tag{1}$$

where $F(x)$ is the cumulative distribution function of the baseline distribution, and $|x| \leq 1$. By putting $\lambda = 0$, the baseline cumulative (cdf) becomes the transmuted distribution, $G(x)$.

In the literature, several research [3, 10] have used the quadratic transmuted distributions and have developed new members of this family of distributions for various choices of baseline cdf $F(x)$. Tahir & Cordeiro [16] have detailed a list of quadratic transmuted distributions.

Granzotto et al. [8] developed a cubic transmuted family of the form:

$$G(x) = \lambda_1 F(x) + (\lambda_2 - \lambda_1)F^2(x) + (1-\lambda_2)F^3(x) \tag{2}$$

with $\lambda_1 \in [0,1]$ and $\lambda_2 \in [-1,1]$, as an extension of the quadratic transmuted distribution.

Several authors [14, 5, 2] have studied various forms of the cubic rank transmuted distributions. In this paper, we propose what we call a quartic rank transmutation map, which is developed in Section 2. In Section 3, the new transmuted map is applied to the exponential distribution to obtain the quartic transmuted exponential distribution (QTED). Relevant characteristics of the new distribution are further developed in the same section. Parameter estimation is carried out in Section 4. In Section 5, applications are implemented using both simulation and real data. Conclusion are drawn in Section 6.

2. Quartic Rank Transmutation Map

In this section, we develop a quartic rank transmutation map. This map is stated in Proposition 2.1.

Proposition 2.1

Let X be a random variable with cdf $F(X)$. Then the quartic rank transmutation map is given by

$$G(x) = F(x)\Big[4\lambda_1 + 6(\lambda_2-\lambda_1)F(x) + 4(\lambda_1 - 2\lambda_2 + \lambda_3)F(x)^2 + (1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3)F(x)^3\Big]$$

with $\lambda_i \in [0,1]$.

Proof:

Let $X_1, X_2, X_3,$ and X_4 be independent and identically distributed random variables distributed with cdf $F(x)$. From the order statistics, we know that:

$$X_{1:4} = \min(X_1, X_2, X_3, X_4)$$

$$X_{2:4} = \text{the } 2^{\text{nd}} \text{ smallest of } (X_1, X_2, X_3, X_4)$$

$$X_{3:4} = \text{the } 3^{\text{rd}} \text{ smallest of } (X_1, X_2, X_3, X_4)$$

$$X_{4:4} = \max(X_1, X_2, X_3, X_4)$$

The cumulative distribution functions of these order statistics are given as follows:

$$F(X_{1:4}) = [1-(1-F(x))]^4$$

$$F(X_{2:4}) = \sum_{i=2}^{4}\binom{4}{i}F(x)^i[1-F(x)]^{4-i}$$

$$F(X_{3:4}) = \sum_{i=3}^{4}\binom{4}{i}F(x)^i[1-F(x)]^{4-i}$$

$$F(X_{4:4}) = [F(x)]^4$$

Now, let

$$G_Y(x) = \begin{cases} Y \stackrel{\mathrm{d}}{=} X_{1:4}, & \text{with probability } \lambda_1 \\ Y \stackrel{\mathrm{d}}{=} X_{2:4}, & \text{with probability } \lambda_2 \\ Y \stackrel{\mathrm{d}}{=} X_{3:4}, & \text{with probability } \lambda_3 \\ Y \stackrel{\mathrm{d}}{=} X_{4:4}, & \text{with probability } \lambda_4 \end{cases}$$

It follows that

$$\sum_{i=1}^{4}\lambda_i = 1 \implies \lambda_4 = 1 - \lambda_1 - \lambda_2 - \lambda_3$$

Then, the distribution of $G_Y(x)$ is given by

$$\begin{aligned} G_Y(x) &= \lambda_1 P(X_{1:4}) + \lambda_2 P(X_{2:4}) + \lambda_3 P(X_{3:4}) + \lambda_4 P(X_{4:4}) \\ &= \lambda_1[1-(1-F(x))]^4 + \lambda_2\sum_{i=2}^{4}\binom{4}{i}F(x)^i[1-F(x)]^{4-i} + \lambda_3\sum_{i=3}^{4}\binom{4}{i}F(x)^i[1-F(x)]^{4-i} + \lambda_4[F(x)]^4 \end{aligned}$$

Expanding, we have

$$\begin{aligned} G(x) = 4\lambda_1 F(x) - 6\lambda_1 F(x)^2 + 4\lambda_1 F(x)^3 - \lambda_1 F(x)^4 6\lambda_2 F(x)^2 - 8\lambda_2 F(x)^3 + 3\lambda_2 F(x)^4 \\ + 4\lambda_3 F(x)^3 - 3\lambda_3 F(x)^4 + F(x)^4 - \lambda_1 F(x)^4 - \lambda_2 F(x)^4 - \lambda_3 F(x)^4 \end{aligned}$$

Now, simplifying, we obtain

$$G(x) = F(x)\Big[4\lambda_1 + 6(\lambda_2 - \lambda_1)F(x) + 4(\lambda_1 - 2\lambda_2 + \lambda_3)F(x)^2 + (1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3)F(x)^3\Big] \tag{3}$$

The corresponding pdf of the quartic rank transmutation map is given by:

$$g(x) = f(x)\Big[4\lambda_1 + 12(\lambda_2 - \lambda_1)F(x) + 12(\lambda_1 - 2\lambda_2 + \lambda_3)[F(x)]^2 + 4(1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3)[F(x)]^3\Big] \tag{4}$$

Equations (3) and (4) would be used as the generators to develop cdf and pdf of quartic transmuted distributions.

3. Quartic Rank Transmuted Exponential Distribution

In this section, we derive the quartic transmuted exponential distribution (QTED) and identify its characteristics.
A random variable X with exponential distribution has the pdf:

$$f(x) = \theta e^{-\theta x}, \qquad x \geq 0, \qquad \theta \geq 0 \tag{5}$$

The corresponding cumulative distribution function (cdf) is given as

$$F(x) = 1 - e^{-\theta x}, \quad x \geq 0, \qquad \theta \geq 0 \tag{6}$$

Let X be a random variable with QTED having cdf $G(X)$. Using Equations (3) and (6), the cdf of the QTED is given as follows:

$$\begin{aligned} G(x) = \left(1 - e^{-\theta x}\right)\Big[&4\lambda_1 + 6(\lambda_2 - \lambda_1)\left(1 - e^{-\theta x}\right) + 4(\lambda_1 - 2\lambda_2 + \lambda_3)\left(1 - e^{-\theta x}\right)^2 \\ &+ (1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3)\left(1 - e^{-\theta x}\right)^3\Big] \end{aligned} \tag{7}$$

The corresponding pdf using Equations (4), (5), and (6) is given as:

$$\begin{aligned} g(x) = \theta e^{-\theta x}\Big[&4\lambda_1 + 12(\lambda_2 - \lambda_1)\left(1 - e^{-\theta x}\right) + 12(\lambda_1 - 2\lambda_2 + \lambda_3)\left(1 - e^{-\theta x}\right)^2 \\ &+ 4(1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3)\left(1 - e^{-\theta x}\right)^3\Big] \end{aligned} \tag{8}$$

where $\theta > 0$ and $\lambda_i \in [0, 1]$.

It can be verified that the pdf in Equation (8) is legitimate. That is, $g(x) \geq 0$ for all x and the integral over $(0, \infty)$ is equal to 1.

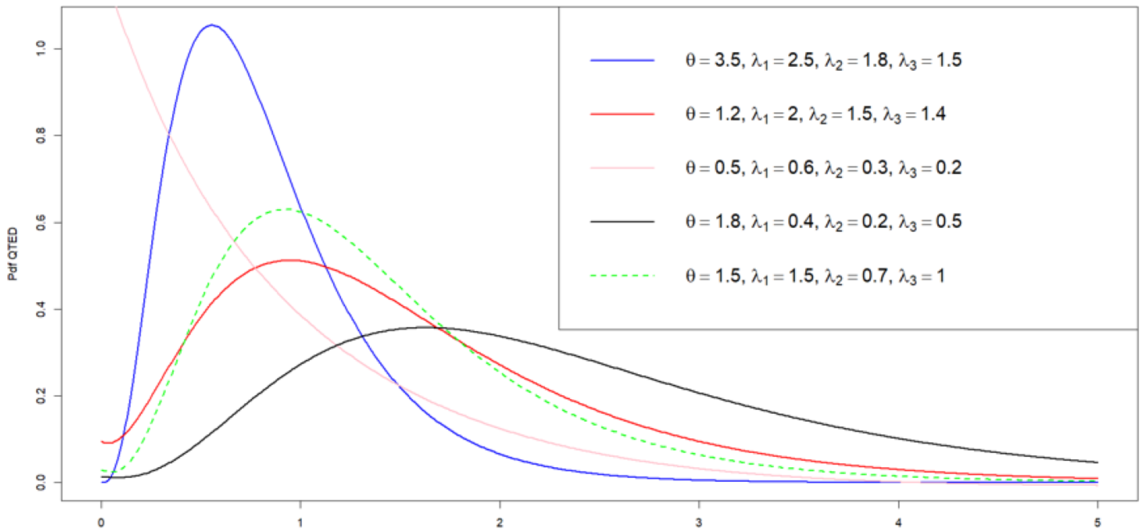

Figure 1. *Pdf of the QTED plotted for various parameter values*

Figure 1 shows the graph of the pdf of QTED for various sets of values of parameters. The graph shows varying shapes depending on the values of the parameters. It can be seen that changing the parameter values alters the characteristics of the distribution, including its center, spread, skewness, and kurtosis.

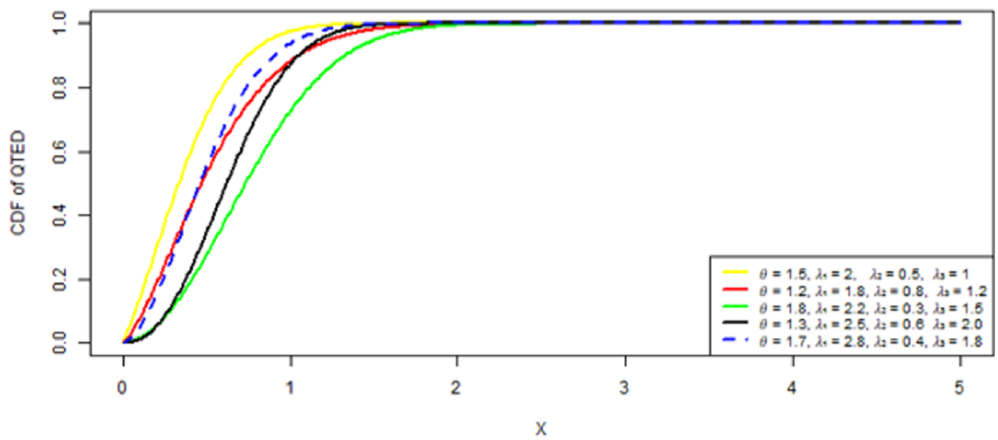

Figure 2. *Cdf of the QTED plotted for various parameter values*

Figure 2 shows the cumulation distribution of the QTED and can be seen that the QTED conforms to a typical distribution function.

3.1. Survival Quantities of the QTED

We explore various survival-related functions for the QTED. These functions provide valuable insights into the behaviour and characteristics of the distribution. Thus, we consider the reliability (survival) and hazard rate functions of the QTED.

3.1.1. Reliability Function of the QTED

Let X be a QTED random variable with cdf $G(x)$. Then the reliability (survival) function of X is given by

$$\begin{aligned} R(x) &= 1 - G(x) \\ &= 1 - \left(1 - e^{-\theta x}\right)\Big[4\lambda_1 + 6(\lambda_2 - \lambda_1)\left(1 - e^{-\theta x}\right) + 4(\lambda_1 - 2\lambda_2 + \lambda_3)\left(1 - e^{-\theta x}\right)^2 \\ &\quad + (1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3)\left(1 - e^{-\theta x}\right)^3\Big] \end{aligned}$$

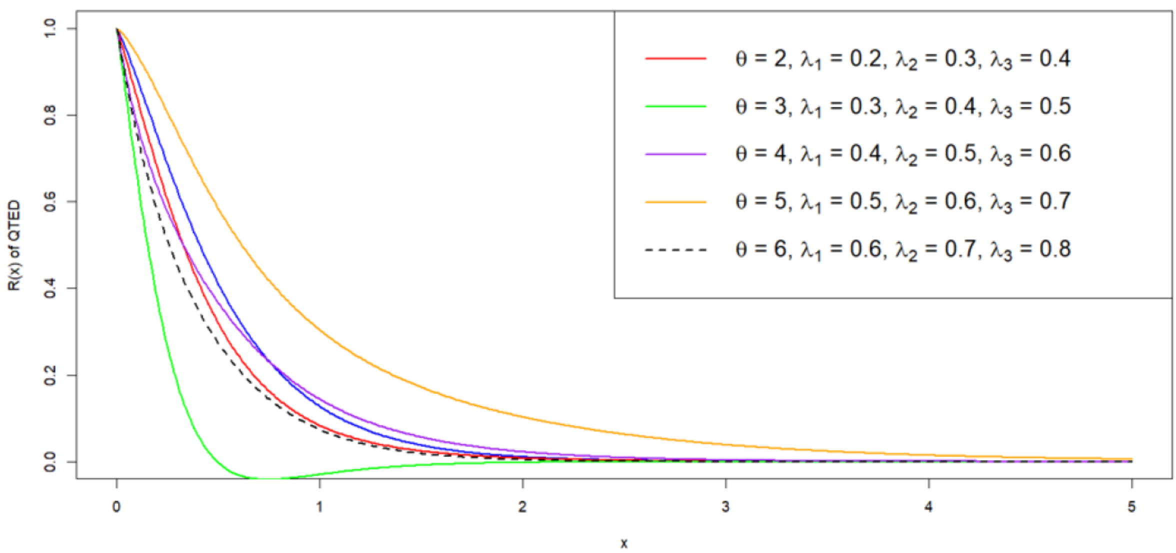

Figure 3. *Survival function of the QTED plotted for various parameter values*

Figure 3 is the plot of the survival function of the QTED for various sets of parameter values. It can be seen that the survival curves show a sharp decline for low values of θ and low values of $\lambda_i(\leq 0.5)$.These indicate a high rate of occurrence of the event or failure. On the other hand, curves show a gradual decline for large values of θ and moderate values of λ_i $(0.5 \leq \lambda_i \leq 0.7)$ suggesting lower rate of occurrence of the event or longer survival time.

3.1.2. Hazard Function of the QTED

The hazard rate for the QTED is given by

$$\begin{aligned}
h(x) &= \frac{g(x)}{1-G(x)} \\
&= \frac{\theta e^{-\theta x}\left[4\lambda_1 + 12(\lambda_2-\lambda_1)(1-e^{-\theta x}) + 12(\lambda_1-2\lambda_2+\lambda_3)(1-e^{-\theta x})^2 + 4(1-2\lambda_1+2\lambda_2-4\lambda_3)(1-e^{-\theta x})^3\right]}{1-\{(1-e^{-\theta x})\left[4\lambda_1+6(\lambda_2-\lambda_1)(1-e^{-\theta x})+4(\lambda_1-2\lambda_2+\lambda_3)(1-e^{-\theta x})^2+(1-2\lambda_1+2\lambda_2-4\lambda_3)(1-e^{-\theta x})^3\right]\}}
\end{aligned}$$

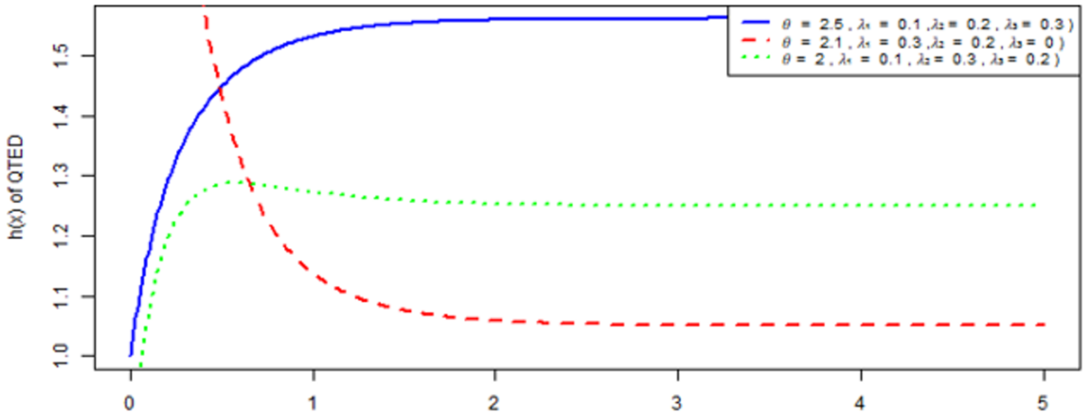

Figure 4. *Hazard function of the QTED plotted for various parameter values*

Figure 4 shows the plot of the hazard function of the QTED for various sets of parameter values.The graph shows an increasing, decreasing, and constant hazard curve, all plotted for different values of the model parameters. Consistent with Figure 3, the hazard curves increase for low values of θ and lower values of $\lambda_i(\leq 0.5), \forall i$. Thus, in this case, the risk of the event occurring increases over time. For low values of θ and low values of λ_i for $i = 3$, the hazard is either constant or declining which shows a low risk of event occurrence over time.

3.2. The rth Moment of the QTED

Let X be a QTED random variable with pdf $g(x)$ as defined in Equation (8). The rth moment of the QTED can be obtained using the relation

$$E(X^r) = \mu_r = \int_{-\infty}^{\infty} x^r g(x) dx \tag{9}$$

Putting Equation (8) into Equation (9), we have

$$\begin{aligned}
E(X^r) = \int_{-\infty}^{\infty} x^r \Bigg\{ \theta e^{-\theta x} \times \Bigg[& 4\lambda_1 + 12(\lambda_2-\lambda_1)(1-e^{-\theta x}) + 12(\lambda_1-2\lambda_2+\lambda_3)(1-e^{-\theta x})^2 \\
& + 4(1-2\lambda_1+2\lambda_2-4\lambda_3)(1-e^{-\theta x})^3 \Bigg] \Bigg\} dx
\end{aligned}$$

By integrating term by term, we have

$$\begin{aligned}
E(X^r) =& 4\lambda_1\theta \int_0^{\infty} x^r e^{-\theta x}\, dx + 12\theta(\lambda_2-\lambda_1) \int_0^{\infty} x^r e^{-\theta x}(1-e^{-\theta x})\, dx \\
& + 12\theta(\lambda_1-2\lambda_2+\lambda_3) \int_0^{\infty} x^r e^{-\theta x}(1-e^{-\theta x})^2\, dx \\
& + 4\theta(1-2\lambda_1+2\lambda_2-4\lambda_3) \int_0^{\infty} x^r e^{-\theta x}(1-e^{-\theta x})^3\, dx
\end{aligned}$$

This can be reduced to

$$E(X^r) = \left[\frac{(-4\lambda_3 + 2\lambda_2 - 2\lambda_1 + 1)}{4^r} + \frac{4(3\lambda_3 + \lambda_1 - 1)}{3^r} + 4(\lambda_3 + \lambda_2 + \lambda_1 - 1) - 3(2\lambda_3 + \lambda_2 + \lambda_1 - 1)2^{1-r}\right]\frac{\Gamma(r+1)}{\theta^r}$$

By setting $r = 1$ and $r = 2$, the following characteristics are obtained respectively.

$$E(X) = \frac{22\lambda_1 + 18\lambda_2 + 12\lambda_3 - 25}{12\theta}$$

$$E(X^2) = \frac{406\lambda_1 + 378\lambda_2 + 300\lambda_3 - 415}{72\theta^2}$$

Subsequently, the variance $Var(X)$, is obtained.

3.3. Order Statistics

Let $X_1, X_2, ..., X_n$ be a random sample from QTED with pdf $g(x)$ and $X_{(1)}, X_{(2)}, ..., X_{(n)}$ the order statistics of the sample, where $X_{(1)} \leq X_{(2)} \leq ... \leq X_{(n)}$, $X_{(1)} = min(X_1, X_2, ..., X_n)$ and $X_{(n)} = max(X_1, X_2, ..., X_n)$.
Then the pdf of the kth order statistic, $X_{(k)}$ is given by

$$f_{k:n}(x) = \frac{n!}{(k-1)!(n-k)!}[G(x)]^{(k-1)}[1 - G(x)]^{(n-k)} \times g(x)$$

Substituting $G(x)$ and $g(x)$, we have

$$\begin{aligned} f_{k:n}(x) = \frac{n!}{(k-1)!(n-k)!} \times &\left\{\theta e^{-\theta x}\left[4\lambda_1 + 12(\lambda_2 - \lambda_1)(1 - e^{-\theta x}) + 12(\lambda_1 - 2\lambda_2 + \lambda_3)(1 - e^{-\theta x})^2\right.\right. \\ &\left.\left. + 4(1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3)(1 - e^{-\theta x})^3\right]\right\} \times \left\{(1 - e^{-\theta x})\left[4\lambda_1 + 6(\lambda_2 - \lambda_1)(1 - e^{-\theta x})\right.\right. \\ &\left.\left. + 4(\lambda_1 - 2\lambda_2 + \lambda_3)(1 - e^{-\theta x})^2 + (1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3)(1 - e^{-\theta x})^3\right]\right\}^{(k-1)} \\ &\times \left\{1 - (1 - e^{-\theta x})\left[4\lambda_1 + 6(\lambda_2 - \lambda_1)(1 - e^{-\theta x}) + 4(\lambda_1 - 2\lambda_2 + \lambda_3)(1 - e^{-\theta x})^2\right.\right. \\ &\left.\left. + (1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3)(1 - e^{-\theta x})^3\right]\right\}^{(n-k)} \end{aligned}$$

Therefore, by putting $k = 1$, the distribution of the minimum order statistic for the QTED is given by

$$\begin{aligned} f_{1:n}(x) = n \times \theta e^{-\theta x} &\left\{\left[4\lambda_1 + 12(\lambda_2 - \lambda_1)(1 - e^{-\theta x}) + 12(\lambda_1 - 2\lambda_2 + \lambda_3)(1 - e^{-\theta x})^2\right.\right. \\ &\left.\left. + 4(1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3)(1 - e^{-\theta x})^3\right]\right\} \times \left\{1 - (1 - e^{-\theta x})\left[4\lambda_1 + 6(\lambda_2 - \lambda_1)(1 - e^{-\theta x})\right.\right. \\ &\left.\left. + 4(\lambda_1 - 2\lambda_2 + \lambda_3)(1 - e^{-\theta x})^2 + (1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3)(1 - e^{-\theta x})^3\right]\right\}^{(n-1)} \end{aligned}$$

Similarly, by putting $k = n$, the maximum order statistic distribution for the QTED can be expressed as

$$f_{n:n}(x) = n\theta e^{-\theta x} \times \left\{ \Big[4\lambda_1 + 12(\lambda_2 - \lambda_1)(1 - e^{-\theta x}) + 12(\lambda_1 - 2\lambda_2 + \lambda_3)(1 - e^{-\theta x})^2 + 4(1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3)(1 - e^{-\theta x})^3 \Big] \right\} \times \left\{ (1 - e^{-\theta x}) \Big[4\lambda_1 + 6(\lambda_2 - \lambda_1)(1 - e^{-\theta x}) + 4(\lambda_1 - 2\lambda_2 + \lambda_3)(1 - e^{-\theta x})^2 + (1 - \lambda_1 + 2\lambda_2 - 4\lambda_3)(1 - e^{-\theta x})^3 \Big] \right\}^{(n-1)}$$

4. Maximum Likelihood Estimation of Parameters of the QTED

The likelihood function for a sample of n observations $x_1, x_2, ..., x_n$ assumed to be taken from QTED with pdf $g(x)$ is given by

$$L(x_1, x_2, ..., x_n | \theta, \lambda_1, \lambda_2, \lambda_3) = \prod_{i=1}^{n} g(x_i; \theta, \lambda_1, \lambda_2, \lambda_3)$$

Let $L(x_1, x_2, ..., x_n | \theta, \lambda_1, \lambda_2, \lambda_3)$ be denoted as $L(X|\theta, \Lambda)$

$$L(X|\theta, \Lambda) = \prod_{i=1}^{n} \left\{ \theta e^{-\theta x_i} \times \Big[4\lambda_1 + 12(\lambda_2 - \lambda_1)(1 - e^{-\theta x_i}) + 12(\lambda_1 - 2\lambda_2 + \lambda_3)(1 - e^{-\theta x_i})^2 + 4(1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3)(1 - e^{-\theta x_i})^3 \Big] \right\}$$

The log-likelihood functions are $l_i = log(L_i)$ and the likelihood equations are

$$l(x_1, x_2, \ldots, x_n | \theta, \lambda_1, \lambda_2, \lambda_3) = \sum_{i=1}^{n} \ln[g(x)]$$

$$l(X|\theta, \Lambda) = \sum_{i=1}^{n} \Big[\ln\theta - \theta x_i + \ln\big(4\lambda_1 + 12(\lambda_2 - \lambda_1)(1 - e^{-\theta x_i}) + 12(\lambda_1 - 2\lambda_2 + \lambda_3)(1 - e^{-\theta x_i})^2 + 4(1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3)(1 - e^{-\theta x_i})^3\big) \Big]$$

Taking the partial derivative of the log-likelihood function with respect to $\theta, \lambda_1, \lambda_2$, and λ_3 and setting to zero gives $\frac{\partial l_i}{\partial \theta} = 0$, $\frac{\partial l_i}{\partial \lambda_1} = 0$, $\frac{\partial l_i}{\partial \lambda_2} = 0$, and $\frac{\partial l_i}{\partial \lambda_3} = 0$
The equations may be solved numerically to obtain the maximum likelihood estimates (MLEs). The R software is used to estimate the parameters based on available data.

5. Simulation Study and Application

In this section, we conduct simulation study to evaluate the properties of the proposed QTED. Furthermore, the practical value of the proposed distribution is demonstrated by utilizing real dataset.

5.1. Simulation Study

A simulation study is conducted by considering samples of sizes 50, 100, 150, 200, 300, 500, and 800 from the QTED. A total of 1000 random samples are generated for each setup with the parameters fixed as

$$\theta = 1.5,\ \lambda_1 = 0.5,\ \lambda_2 = 0.6,\ \text{and}\ \lambda_3 = 02$$

Table 1 presents the mean of the true value, estimates, bias, and standard error (SE) of the model parameters. From the table, it can be observed that the estimated values of the parameters $\theta, \lambda_1, \lambda_2, \lambda_3$ tend to be close to the true values for larger sample sizes.

Table 1. *Parameter Estimates of the QTED by Simulation*

Sample size	True value	Estimate	—Bias—	SE
50	$\theta = 1.5$	1.2084	0.2916	0.0412
	$\lambda_1 = 0.5$	0.3148	0.1852	0.0262
	$\lambda_2 = 0.6$	0.4327	0.1673	0.0234
	$\lambda_3 = 0.2$	0.0934	0.1066	0.0151
100	$\theta = 1.5$	1.3172	0.1828	0.0183
	$\lambda_1 = 0.5$	0.3736	0.1264	0.0126
	$\lambda_2 = 0.6$	0.4651	0.1349	0.0135
	$\lambda_3 = 0.2$	0.1018	0.0982	0.0098
150	$\theta = 1.5$	1.3382	0.1618	0.0132
	$\lambda_1 = 0.5$	0.3865	0.1135	0.0093
	$\lambda_2 = 0.6$	0.4651	0.1349	0.0110
	$\lambda_3 = 0.2$	0.1018	0.0982	0.0080
200	$\theta = 1.5$	1.4382	0.0618	0.0044
	$\lambda_1 = 0.5$	0.4144	0.0856	0.0061
	$\lambda_2 = 0.6$	0.5164	0.0836	0.0059
	$\lambda_3 = 0.2$	0.1168	0.0832	0.0058
300	$\theta = 1.5$	1.5382	0.0382	0.0022
	$\lambda_1 = 0.5$	0.4365	0.0635	0.0037
	$\lambda_2 = 0.6$	0.5656	0.0344	0.0020
	$\lambda_3 = 0.2$	0.1450	0.0550	0.0031
500	$\theta = 1.5$	1.4823	0.0177	0.0007
	$\lambda_1 = 0.5$	0.5214	0.0214	0.0009
	$\lambda_2 = 0.6$	0.6252	0.0252	0.0011
	$\lambda_3 = 0.2$	0.1837	0.0163	0.0007
800	$\theta = 1.5$	1.5138	0.0138	0.0005
	$\lambda_1 = 0.5$	0.5140	0.0140	0.0005
	$\lambda_2 = 0.6$	0.6179	0.0179	0.0006
	$\lambda_3 = 0.2$	0.1889	0.0111	0.0004

Overall, the results indicate that increasing the sample size improves the accuracy and precision of parameter estimation as the estimates yield lower bias and SE.

5.2. Application to Life Test Data

In this part, we demonstrate the implementation of the proposed distribution to real-life data. The data (given in Table 2) contains the times to failure of 50 devices put on life test at time 0 (in weeks) which is extracted from [1].

Table 2. *Lifetimes of 50 devices*

0.1	0.2	1	1	1	1	1	2	3	6
7	11	12	18	18	18	18	18	21	32
36	40	45	46	47	50	55	60	63	63
67	67	67	67	72	75	79	82	82	83
84	84	84	85	85	85	85	85	86	86

The descriptive statistics of the data are shown in Table 3 below.

Table 3. *Descriptive statistics of the lifetime data*

Min	1^{st} Qu.	Median	Mean	3^{rd} Qu.	Max	Var	Std. Dv.	Skew	Kurt
0.1	13.50	48.50	45.69	81.25	86	1078.153	32.835	-0.138	1.414

From the table, the distribution is negatively skewed suggesting a concentration of shorter lifetimes, but devices generally exhibit longer lifetimes. Also, the standard deviation shows that there is a large variability in device lifetimes, indicating potential inconsistencies in performance. Meanwhile, a positive kurtosis implies slightly heavier tails in the distribution, potentially indicating some devices with exceptionally long lifetimes, contributing to a more peaked distribution.

Figure 5 shows the statistical plots for the lifetime data.

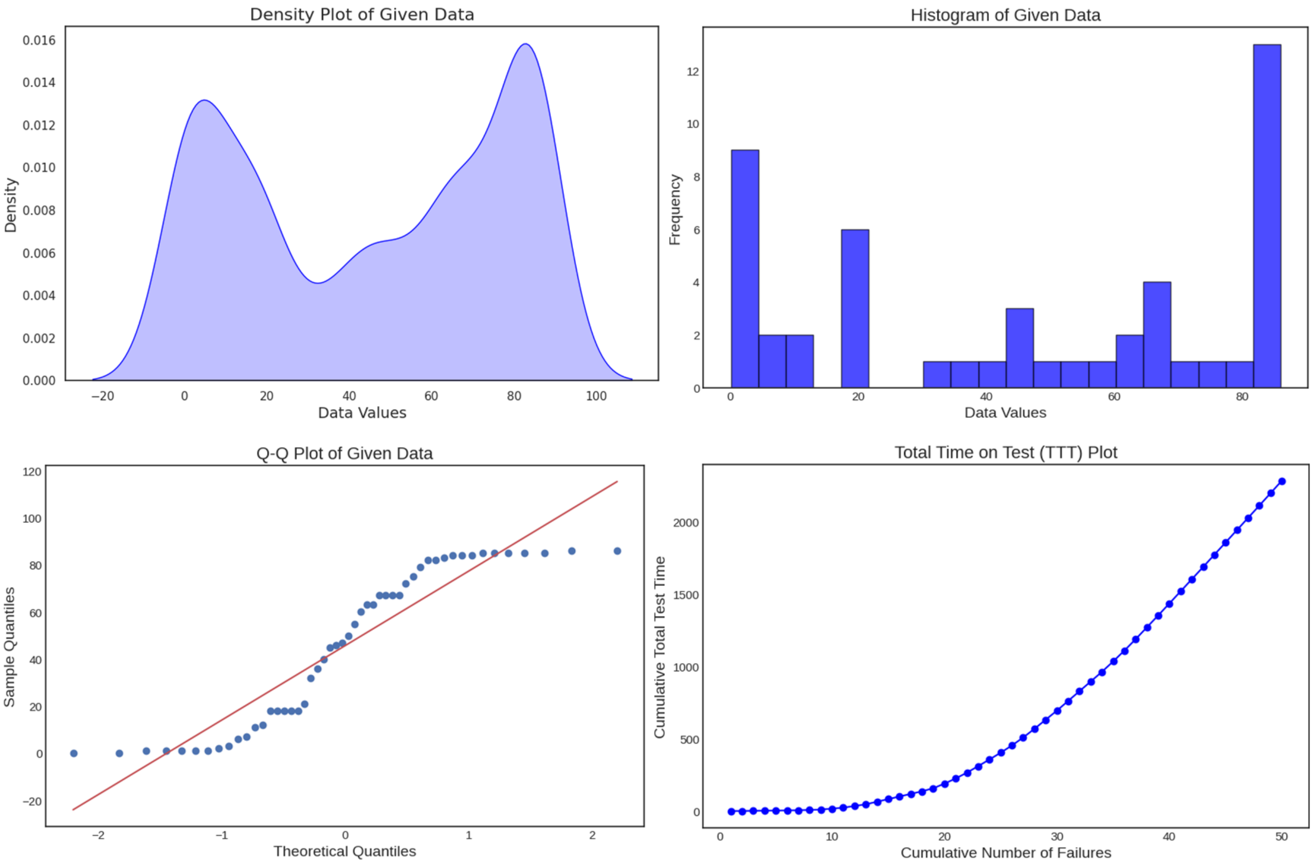

Figure 5. *Graphical plots based on the data*

The density graph shows that the data may be underlined by multiple distributions and hence has some complexities. The Q-Q plot compares the quantiles of the data to the quantiles of the theoretical distribution. The Q-Q plot shows that the upper quantiles of the data are larger than those of the theoretical distribution, while the lower quartiles are smaller. Furthermore, this is also an indication that there may be extreme values on the right side of the distribution. The TTT plot also suggests that a proportion of devices experience the event later than expected.

The goodness of fit of the QTED is compared with the following distributions:

1. The exponential distribution given in Equation (5)
2. Cubic Transmuted Exponential (CTED) [13], which given as

$$f(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}} \Bigg[1 + \lambda_1 + 2(\lambda_2 - \lambda_1)(1 - e^{-\frac{x}{\theta}}) + 3\lambda_2 (1 - e^{-\frac{x}{\theta}})^2 \Bigg]$$

   where $\lambda_1, \lambda_2 \in [-1, 1]$, $\theta \in (0, \infty]$, such that $-2 \leq \lambda_1 + \lambda_2 \leq 1$ and $x \in (0, \infty)$
3. Transmuted Exponential Distribution (TED) [11], which is given as

$$f(x) = \frac{1}{\theta} \exp\left(-\frac{x}{\theta}\right) \left[1 - \lambda + 2(\lambda) \exp\left(-\frac{x}{\theta}\right)\right]$$

   where $x > 0,\ \theta > 0$, and $|\lambda \leq 1|$

Table 4 presents the ML estimates of the parameters of three different transmuted probability distributions (QTED, CTED, TED) and the exponential distribution.

Table 4. *MLEs of selected distributions*

Distribution	Parameter	Estimate
QTED	$\theta, \lambda_1, \lambda_2, \lambda_3$	0.239, 0.554, 0.674, 0.211
CTED	$\theta, \lambda_1, \lambda_2$	33.77, -0.064, -0.27
TED	θ, λ	41.157, -0.243
Exponential	θ	27.36

These figures represent the values obtained through the maximum likelihood estimation process for each distribution and parameter, indicating the best-fit values for the given dataset. They provide insights into the characteristics of the distributions and their relationship with the observed data.

Table 5. Selection criteria values for selected models

Distribution	-LogLik	AIC	AICc	BIC
QTED	197.794	397.588	394.476	405.236
CTED	236.018	478.036	478.557	483.772
TED	240.677	485.355	485.610	489.179
Exponential	257.780	517.560	517.861	522.440

Table 5 presents selection criteria values for selected models, which include the log-likelihood, AIC (Akaike Information Criterion), AICc (corrected AIC), and BIC (Bayesian Information Criterion). It can be seen that the QTED consistently yields the smallest values and therefore provides the best fit for the data.

6. Conclusion

A quartic rank transmutation map that is proposed in this paper has been applied to develop a new transmuted exponential distribution (QTED). The new distribution is found to provide the most suitable fit for data with underlying complexity compared to the existing quadratic and cubic transmuted exponential distributions. The characteristics and performance of the new QTED are demonstrated and could be useful for obtaining new transmutations of other relevant distributions.

Abbreviations

AIC	Akaike's Information Criterion
AICc	Corrected Akaike's Information Criterion
BIC	Bayesian Information Criterion
CTED	Cubic Transmuted Exponential Distribution
MLE	Maximum Likelihood Estimation
TE	Transmuted Exponential
QTED	Quartic Transmuted Exponential Distribution
QTRD	Quartic Rank Transmutation Distribution
SE	Standard Error

Conflict of Interest

The authors declare no conflicts of interest.

References

[1] Aarset, M. V. (1987). How to identify bathtub hazard rate. *IEEE Transactions on Reliability*, 36: 106-108, https://doi.org/10.1109/TR.1987.5222310

[2] Al-Kadim, K. A., & Mohammed, M. H. (2017). The cubic transmuted Weibull distribution. *Journal of University of Babylon*, 3, 862-876.

[3] Aryal, G. R. & Tsokos, C. P. (2009). On the transmuted extreme value distribution with application. Nonlinear Analysis: *Theory, Methods and Applications*, 71: 1401-1407, https://doi.org/10.1016/j.na.2009.01.168

[4] Burr, I. W. (1942). Cumulative frequency functions. *Annals of Mathematical Statistics*, 13: 215-232, https://doi.org/10.1214/aoms/1177731607

[5] Celik, N. (2018). Some cubic rank transmuted distributions, *Journal of Applied Mathematics, Statistics and Informatics* 14(2), 27-43.

[6] Cordeiro, G. M. & Castro, M. de. (2011). A new family of generalized distributions. *Journal of Statistical Computation and Simulation*, 81: 883-898.

[7] Eugene, N., Lee, C. & Famoye, F., (2002). Beta-normal distribution and its applications. *Communications in Statistics-Theory and Methods*, 31: 497-512.

[8] Granzotto, D. C. T., Louzada, F., & Balakrishnan, N. (2017). Cubic rank transmuted distributions: inferential issues and applications. *Journal of Statistical Computation and Simulation*, 87(14), 2760-2778. https://doi.org/10.1080/00949655.2017.1344239

[9] Marshall, A. W. & Olkin, I. (1997). A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families. *Biometrika*, 84: 641-652, https://doi.org/10.1093/biomet/84.3.641

[10] Merovci, F. & Puka, L. (2014). Transmuted Pareto Distribution. *ProbStat Forum* 7: 1-11.

[11] Owoloko, E. A., Oguntunde, P.E. & Adejumo, A. O. (2015). Performance rating of the transmuted exponential distribution: An analytical approach. *Springer Plus*, 4: 8-18, https://doi.org/10.1186/s40064-015-1590-6

[12] Pearson, K. (1895). Contributions to the mathematical theory of evolution, II: Skew variation in homogeneous material. *Philosophical Transactions of the Royal Society*, 186: 343-414, https://doi.org/10.1098/rsta.1895.0010

[13] Rahman, M. M., Al-Zahrani, B., & Shahbaz, M. Q. (2018a). Cubic Transmuted Pareto Distribution. *Annals of Data Science*, 1-18.

[14] Rahman, M. M., Al-Zahrani, B., & Shahbaz, M. Q. (2018). A General Transmuted Family of Distributions. *Pak J Stat Oper Res*, 14: 451-469, https://doi.org/10.18187/pjsor.v14i2.2334

[15] Shaw, W. T. & Buckley, I. R. C. (2009). The alchemy of probability distributions: beyond Gram-Charlier expansions, and a skew-kurtotic-normal distribution from a rank transmutation map. Research report.

[16] Tahir, M. H. & Cordeiro, G. M. (2016). Compounding of distributions: a survey and new generalized classes. *Journal of Statistical Distributions and Applications*, 3, 1-35.